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# On the Long Time Behavior of a Particle in an Infinitely Extended System in One Dimension[1]

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We study the long time behavior of a non-equilibrium infinite particle system in one dimension. First, we show that the velocity of a particle increases at most linearly in time. Then we discuss at a heuristic level the displacement of a particle when the mutual interaction is singular. Finally we study the motion of a fast particle interacting with a background of slow particles and we prove that the velocity of the fast particle remains almost unchanged for a long time (at least proportional to the velocity itself).

**KEY WORDS:** Infinite dynamics; long time behavior.

## 1. INTRODUCTION

In the present paper we study the long time behavior of a non-equilibrium particle system in one dimension. In particular we investigate three different problems. We consider a system of infinitely many particles interacting via Newton's law. A phase point of the system is an infinite sequence $X = \{x_i, v_i\}_{i \in \mathbb{N}}$ of positions and velocities and its time evolution is characterized by the solutions of the Newton equations:

$$\ddot{x}_i(t) = \sum_{\substack{j \in \mathbb{N} \\ j \neq i}} F(x_i(t) - x_j(t)), \qquad i \in \mathbb{N}, \tag{1.1}$$

where $F(x) = -\nabla\phi(x)$ and $\phi$ is a two-body potential. Equation (1.1) are complemented by the initial conditions $\{x_i(0), v_i(0)\}_{i \in \mathbb{N}}$. The initial conditions

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are chosen in a set sufficiently large to be the support of the states of interest from a thermodynamical point of view.

We consider a tagged particle and we follow its time evolution. Obviously its velocity may grow or decrease during the motion in a complicated way. Here we want to find a bound on it. Actually the time evolution of similar systems have been studied in the literature in many papers (see refs. 1–7 and papers quoted there in) as the first step to a rigorous study of Nonequilibrium Statistical Mechanics. An essential tool to prove the existence of this evolution consists in the control of the velocity of each particle in a way which prevents the built up of large densities. Unfortunately the bounds used in these papers are very bad for large times. In the present paper we want to show a simple bound on the growth of the velocity that can be obtained by using some unsophisticated (but not trivial) arguments and that remains significant also for large times.The proof consists of an application in one dimension of some estimates developed by Dobrushin and Fritz in refs. 2 and 3, that will be used in the form discussed in ref. 1.

The exact statement of the result and the proofs are given in Section 2. In Section 3 we discuss, as a second problem, at a heuristic level the displacement of a particle when the two body potential is singular. In Section 4 we study the motion of a fast particle interacting with a background of slow particles. We prove that the background cannot slow down rapidly the fast particle, that conserves its velocity for a very long time. Finally in the Appendix we report a technical Lemma.

## 2. BOUNDS ON THE GROWTH OF THE VELOCITY OF A PARTICLE

We consider the system (1.1) and assume that the particles interact by means of a not negative, short-range, bounded, twice differentiable, two-body potential $\phi = \phi(|x|)$, $x \in \mathbb{R}$:

$$\phi(0) > 0, \quad \phi(|x|) = 0 \qquad \text{if} \quad |x| \geqslant r > 0. \tag{2.2}$$

We believe that the results could be extended to more general interactions, but with more technical efforts. A generalization will be shortly discussed at the end of this section.

We consider initial data $X$ with a locally finite density and energy. In order to consider configurations which are typical for thermodynamical states, we must allow logarithmic divergences in the velocities and local densities. More precisely, define

$$Q(X) = \sup_{\mu} \sup_{R:\, R > \log(e+|\mu|)} \frac{Q(X;\mu, R)}{2R} \tag{2.3}$$

where

$$Q(X;\mu,R)=\sum_i \chi(|x_i-\mu|\leqslant R)\left(\frac{v_i^2}{2}+\frac{1}{2}\sum_{j:j\neq i}\phi_{i,j}+1\right) \tag{2.4}$$

where here and in the sequel $\chi(A)$ will indicate the characteristic function of the set $A$, and $\phi_{i,j}=\phi(|x_i-x_j|)$.

We consider initial data for which $Q(X)<\infty$. It has been shown that this set has a full measure with respect to any Gibbs state (see ref. 2).

We state now the main result of the present section:

**Theorem 2.1.** For any fixed $R^*>0$, we consider the particles such that $|x_i(0)|<R^*$. Then there exist two positive constants $C_1$, $C_2$ ($C_1$ depending on $R^*$ and $C_2$ independent of it), such that for these particles

$$|v_i(t)|\leqslant C_1+C_2t; \qquad t\geqslant 0. \tag{2.5}$$

We remark that the previous bound implies that the effective force acting (in average) on a very fast particle is bounded by a constant, in spite of the fact that the infinite size of the system can produce, apriori, large concentrations and then large forces.

*Proof.* First we must give sense to the time evolution of a system of infinite equations. To this purpose, the solution of Eq. (1.1) will be constructed by means of a limiting procedure. We define a partial dynamics by neglecting all the particles outside $B(0,n)$, where $B(\mu,R)=\{y\in\mathbb{R}\mid |y-\mu|<R\}$. More precisely we consider, for a positive integer $n$, the differential system:

$$\begin{gathered}\ddot{x}_i(t)=F_i(X^n(t))\\ x_i^n(0)=x_i,\quad v_i^n(0)=v_i,\qquad i\in I_n,\end{gathered} \tag{2.6}$$

where

$$\begin{gathered}I_n=\{i\in\mathbb{N}\mid x_i\in B(0,n)\},\\ F_i(X^n(t))=\sum_{j:j\neq i}F(x_i^n(t)-x_j^n(t))\end{gathered} \tag{2.7}$$

and

$$X^n(t)=\{x_i^n(t),v_i^n(t)\}_{i\in I_n} \tag{2.8}$$

is the time evolved finite configuration. We call it $n$-partial dynamics.

It can be proved (see the quoted literature) that

**Theorem 2.2.** Let $\mathscr{X} = \{X \mid Q(X) < +\infty\}$ and $X \in \mathscr{X}$. Then there exists a unique flow $t \to X(t) = \{x_i(t), v_i(t)\}_{i \in \mathbb{N}} \in \mathscr{X}$ satisfying:

$$\ddot{x}_i(t) = F_i(X(t)), \qquad X(0) = X. \tag{2.9}$$

Moreover, for all $t \in \mathbb{R}$ and $i \in \mathbb{N}$

$$\lim_{n \to \infty} x_i^n(t) = x_i(t), \qquad \lim_{n \to \infty} v_i^n(t) = v_i(t). \tag{2.10}$$

Now we prove the main result. The strategy is the following: we obtain a bound like (2.5) for the $n$-partial dynamics and then we extend this result to the infinite dynamics.

We introduce a mollified version of the energy plus the number of the particles contained in the interval $B(\mu,R)$:

$$W(X;\mu,R) = \sum_i f_i^{\mu,R} \left( \frac{v_i^2}{2} + \frac{1}{2} \sum_{j: j \neq i} \phi_{i,j} + 1 \right) \tag{2.11}$$

where:

$$f_i^{\mu,R} = f\left( \frac{|x_i - \mu|}{R} \right) \tag{2.12}$$

and the function $f \in C^\infty(\mathbb{R}^+)$ satisfies:

$$f(x) = 1 \qquad \text{for} \quad x \in [0,1]$$

$$f(x) = 0 \qquad \text{for} \quad x \in (2,+\infty).$$

and $|f'(x)| \leqslant 2$.

We define

$$W(X) = \sup_{\mu} \sup_{R: R > \log(e+|\mu|)} \frac{W(X;\mu,R)}{2R} \tag{2.13}$$

Is is obvious that

$$Q(X;\mu,R) \leqslant W(X;\mu,R) \leqslant Q(X;\mu,2R) \tag{2.14}$$

so that $Q(X(0))$ controls $W(X(0))$ and $W(X(t))$ controls $Q(X(t))$.

The main tool to obtain a bound like (2.5) for the partial dynamics is the following lemma:

**Lemma 2.1.**

$$\sup_{\mu} W(X^n(t); \mu, R(n, t)) \leqslant C_3 R(n, t) \tag{2.15}$$

where

$$R(n, t) = \log(e+n) + \int_0^t ds\, V^n(s) \tag{2.16}$$

and

$$V^n(s) = \max_i \sup_{0 \leqslant \tau \leqslant s} |v_i^n(\tau)| \tag{2.17}$$

From now on $C_i$ denotes a constant independent of $t$ and $n$.

The proof is essentially contained in ref. 1 and we write it in the Appendix for completeness.

As a consequence of Lemma 2.1, we have:

$$Q(X^n(t); \mu, R(n, t)) \leqslant W(X^n(t); \mu, R(n, t)) \leqslant C_3 R(n, t). \tag{2.18}$$

Equation (2.18) allows to control the number $N(n, \mu, t)$ of the particles in $B(\mu, R(n, t))$ having velocity larger than the quantity

$$b(n, t) = C_4(\log(e+n))^{1/2} + \tfrac{1}{2} V^n(t). \tag{2.19}$$

Indeed, by definition of $Q(X^n(t); \mu, R(n, t))$, neglecting some positive terms, we have:

$$Q(X^n(t); \mu, R(n, t)) > \tfrac{1}{2} [b(n, t)]^2\, N(n, \mu, t) \tag{2.20}$$

By the definition of $R(n, t)$ and inequality (2.15):

$$N(n, \mu, t) < \frac{2C_3[\log(e+n) + tV^n(t)]}{[C_4(\log(e+n))^{1/2} + \frac{1}{2} V^n(t)]^2}. \tag{2.21}$$

Neglecting positive terms in the denominator, we obtain:

$$N(n, \mu, t) < \frac{2C_3}{C_4^2} + \frac{8C_3 t}{4C_4 + V^n(t)} \tag{2.22}$$

We observe that $V^n(t)$ large implies $N(n, \mu, t)$ small. We choose $C_4$ such that $\frac{2C_3}{C_4^2} = \frac{1}{4}$ and for $V^n(t) \geqslant 32C_3 t$ from Eq. (2.22) we get $N(n, \mu, t) \leqslant \frac{1}{2}$. Therefore there are no particles in $B(\mu, R(n, t))$ faster than $b(n, t)$. Since the argument is independent of $\mu$, there are no particles faster than $b(n, t)$. Therefore

$$V^n(t) \leqslant \max\{32C_3 t, C_4(\log(e+n))^{1/2} + \tfrac{1}{2} V^n(t)\} \tag{2.23}$$

which implies that there exist two constants $C_5$, $C_6$ such that

$$V^n(t) < C_5(\log(e+n))^{1/2} + C_6 t, \qquad t \geqslant 0. \tag{2.24}$$

Now we must prove that (2.24), valid for $n$-partial dynamics, implies for the infinite dynamics the bound (2.5). To do this, we evaluate the difference between the partial and the infinite dynamics, when they act on a particle such that $|x_i(0)| < R^*$ or, using a shorter notation, $i \in I^*$.

From the equation of motion written in the integral form we have:

$$x_i^n(t) - x_i^{n-1}(t) = \int_0^t ds(v_i^n(s) - v_i^{n-1}(s)) \tag{2.25}$$

$$v_i^n(t) - v_i^{n-1}(t) = \int_0^t ds \left[ \sum_j F(x_i^n(s) - x_j^n(s)) - \sum_j F(x_i^{n-1}(s) - x_j^{n-1}(s)) \right]. \tag{2.26}$$

Define

$$\delta_i(n, t) = |x_i^n(t) - x_i^{n-1}(t)| + |v_i^n(t) - v_i^{n-1}(t)| \tag{2.27}$$

we have, by the Lipschitz property of the force $F$:

$$\delta_i(n, t) \leqslant C_7 \int_0^t ds \sum_j{}^* [\delta_i(n, s) + \delta_j(n, s)] \tag{2.28}$$

where $\sum_j^*$ means the sum restricted to all particles closer than $r$ (the range of the interaction) to $x_i^n(s)$ or $x_i^{n-1}(s)$ for $s \leqslant t$. It is easy to give a bound to this quantity. In fact from (2.24) each particle moves at most as

$$(C_5(\log(e+n))^{1/2} + C_6 t)\, t \tag{2.29}$$

and hence it can interact only with the particles initially contained in an interval of size

$$p(n, t) = 2(C_5(\log(e+n))^{1/2} + C_6 t)\, t + 2r. \tag{2.30}$$

By the definition on $Q(X)$, the number of these particles can be bounded by the quantity

$$g(n, t) = Q(X)[\log(e+n) + p(n, t)]. \tag{2.31}$$

Hence, defining

$$u_k(n, t) = \sup_{i \in I_k} \delta_i(n, t), \tag{2.32}$$

we have that:

$$u_k(n, t) \leqslant C_8\, g(n, t) \int_0^t ds\, u_{k_1}(n, s), \tag{2.33}$$

where

$$k_1 = \mathrm{Int}[k + p(n, t) + 1] \tag{2.34}$$

and Int$[\cdot]$ means integer part of $[\cdot]$.

We iterate (2.33), starting from the interval $[-R^*, R^*]$ and arriving close to the interval $[-(n-1), n-1]$, after $m$ steps of size $p(n, t)$. We choose $n > R^* + 1$. Of course

$$m \geqslant \mathrm{Int}\left[\frac{(n-1) - R^*}{p(n, t)}\right] \tag{2.35}$$

We stop the iterative procedure by means of (2.24) and (2.29). In conclusion

$$u_{k_o}(n, t) < C_9 a(n, t)(C_8\, g(n, t))^m \frac{t^m}{m!} \tag{2.36}$$

where

$$a(n, t) = (C_5(\log(e+n))^{1/2} + C_6 t)(t+1) \tag{2.37}$$

and $k_o$ refers to particles in $I^*$. We choose now a $n^*$-partial dynamics. For this dynamics bound (2.24) holds. We evaluate the difference between the velocity of a particle of $I^*$ moving via the $n^*$-partial dynamics and the infinite one. From (2.36) this difference is bounded by

$$\sum_{n=n^*}^{\infty} u_{ko}(n, t) < D, \tag{2.38}$$

where

$$D = \sum_{n=n^*}^{\infty} C_9 a(n, t)(C_8 g(n, t))^m \frac{t^m}{m!}.$$

We choose

$$n^* = \mathrm{Int}[e^t + 2R^* + 2]. \tag{2.39}$$

Obviously for $n \geqslant n^*$

$$t < \log(e+n) \tag{2.40}$$

$$p(n, t) < C_{10} \log^2(e+n) \tag{2.41}$$

$$g(n, t) < C_{11} \log^2(e+n) \tag{2.42}$$

$$a(n, t) < C_{12} \log^2(e+n) \tag{2.43}$$

so that, by using the Stirling formula $\log(n!) > n(\log(n) - 1)$, and (2.35), (2.39), (2.40)–(2.43) we prove that the series in Eq. (2.38) is convergent and its sum can be easily bounded independently of $t$. Hence from bound (2.24) and (2.38) we have

$$|v_i(t)| < C_5(\log(e+n^*))^{1/2} + C_6 t + D < C_5 \log(e+n^*) + C_6 t + D. \tag{2.44}$$

From the definition of $n^*$ the proof is achieved. ▮

Until now we have study bounded interactions. By a small effort, we can extended Theorem 2.1 to interactions with a powerlike singularity. More precisely the following result holds:

**Theorem 2.3.** Let us consider an interaction produced by a two body potential $\phi = \phi(|x|)$, $x \in \mathbb{R}$, that we assume to be nonnegative and short range, i.e., there exists a positive constant such that

$$\phi(|x|) = 0 \qquad \text{if} \quad |x| > r. \tag{2.45}$$

Moreover the potential is twice differentiable if $|x| > 0$ and there exist two nonnegative constants $a$, $b$ such that

$$\phi(|x|) = \phi_1(|x|) + a\,|x|^{-b} \tag{2.46}$$

where $\phi_1(|x|)$ is twice differentiable.

Then for this interaction the result of Theorem 2.1 holds.

*Proof.* The proof is similar to the previous one and we outline only the two points in which they differ: the proof of Lemma 2.1 and the proof of (2.28) and on, where we have used the Lipschitz property of the force.

Regarding the proof of Lemma 2.1, we observe that the assumption (2.46) implies:

$$|x|\,|F(x)| \leqslant C_{13} + C_{14}\phi(|x|) \tag{2.47}$$

so that from (A.5), after obvious modification in (A.7), we obtain in this case also (A.9) and subsequent ones.

Regarding to the Lipschitz property of the force, we observe that in general

$$|F(x) - F(y)| \leqslant L\,|x-y|. \tag{2.48}$$

In the previous case $L$ was a constant, while here it depends on $\min\{|x|, |y|\}$ and diverges when the minimum goes to zero. But in this limit the energy also diverges, so that the Lipschitz constant can be controlled by a power of the Energy of a region containing $x$ and $y$. This fact implies that in (2.33) a power of $g(n, t)$ comes out, but this does not change the remaining steps of the proof. ▮

## 3. BOUNDS ON THE DISPLACEMENT OF A TAGGED PARTICLE

In this section we discuss at a heuristic level a problem related to the one rigorously studied in Section 2.

We consider a particle initially close to the origin and we study its displacement during the time. Of course its position increases or decreases in a complicate way. We want to give a bound on its maximal displacement, which remains significant for long time also. Obviously the bound obtained in Section 2 on the maximal velocity implies that the displacement increases at most quadratically in time. We discuss now how to improve this result.

We consider a system of particles interacting via a power-like singular interaction as in Theorem 2.3 with $a, b > 0$. The initial data are chosen in a set smaller than the set $\mathscr{X}$ considered in the previous section. Actually a typical configuration of our set belongs to $\mathscr{X}$. Moreover there are not too many holes between the particles, in the sense that there is an $R^*$ large enough such that for any $R \geqslant R^*$ in $B(0, R)$ there are at least const. $R$ particles. It is possible to convince oneself that all sets of thermodynamical relevance enjoy this property. Since the order of the particles is fixed, if a tagged particle has a (large) displacement $|s(t)|$, also $n(t) = \text{const.}\ |s(t)|$ other particles have a displacement of the same order. Of course there is a relation between the displacement $s_i(t)$ of a particle and its change in the kinetic energy $T_i(t) = \frac{1}{2} v_i(t)^2$. Indeed by convexity, we obtain:

$$|s_i(t) - s_i(0)| \leqslant \int_0^t |v_i(s)| \, ds \leqslant (2)^{\frac{1}{2}} (t)^{\frac{1}{2}} \left( \int_0^t T_i(s) \, ds \right)^{\frac{1}{2}}. \tag{3.1}$$

In conclusion the time integral of the energy of a region containing these particles increases during a (large) time $t$ at least as

$$\text{const.} \frac{s^2(t)}{t} |s(t)|. \tag{3.2}$$

This quantity must be compared with the time integral of the energy of the region. We observe that, by Lemma 2.1 and (2.5), the energy is bounded by const. $t^2$. Hence the time integral of this energy increases at most like const. $t^3$ and we obtain (for large $t$)

$$|s(t)| \leqslant \text{const. } t^{4/3}. \tag{3.3}$$

Of course we do not believe that this bound is optimal. However we remark that our initial data contain a rigid translation of the system that trivially gives $s(t) = \text{const. } t$. Hence the exponent of the time dependence $(4/3)$ in (3.3) does not seem too bad.

It would be interesting to discuss some properties of $s(t)$ and not only a bound on it, as it has been done for particular systems (see for instance ref. 8), but it seems too hard.

## 4. INTERACTION OF A FAST PARTICLE WITH A BACKGROUND OF SLOW PARTICLES

We consider a system composed by a tagged particle of position and velocity $(\hat{x}, \hat{v})$ and mass $M$, interacting via a two-body force $\hat{F}(\hat{x}(t) - x_i(t))$ with an infinite particle system like that discussed in Theorem 2.1. The equations of motion read:

$$\begin{aligned} M\ddot{\hat{x}}(t) &= \sum_{j \in \mathbb{N}} \hat{F}(\hat{x}(t) - x_j(t)), \\ \ddot{x}_i(t) &= \sum_{\substack{j \in \mathbb{N} \\ j \neq i}} F(x_i(t) - x_j(t)) + \hat{F}(x_i(t) - \hat{x}(t)), \qquad i \in \mathbb{N}, \end{aligned} \tag{4.1}$$

where $\hat{F}(x) = -\nabla\hat{\phi}(|x|)$, $F(x) = -\nabla\phi(|x|)$ and $\hat{\phi}$, $\phi$ are two-body potentials, that we suppose twice differentiable, positive, with a short range $\hat{r}$, $r$ respectively and $\phi(0) > 0$.

The existence and the uniqueness of the solutions of (4.1) is an easy generalization of Theorem 2.2.

We suppose that initially the fast particle is close to the origin and we denote by $\hat{v}_o > 0$ its initial velocity. We shall prove that the fast particle does not loose its velocity until a time proportional to const. $\hat{v}_o$, i.e., the background cannot slow down the fast particle for a time that becomes very long as $\hat{v}_o$ becomes very large. More precisely:

**Theorem 4.1.** There exist two positive constants $C^+$, $\hat{C}$ such that for any $\hat{v}_o$

$$|\hat{v}(t) - \hat{v}_o| \leqslant C^+ \qquad \text{if} \quad 0 \leqslant t \leqslant \hat{C}\hat{v}_o. \tag{4.2}$$

We outline that $C^+$ and $\hat{C}$ depend on the initial state of the background but they are, obviously, independent of $\hat{v}_o$. Theorem 4.1 becomes significant when $\hat{v}_o$ is large.

*Proof.* For finite $\hat{v}_o$ Theorem 4.1 is a trivial consequence of the existence of the dynamics. Then we assume in the sequel that $\hat{v}_o$ is large enough. From now on we assume $\hat{v}_o \ll a$, where $a$ is the maximum of the modulus of the initial velocity of the particles of the background initially contained in $B(0, 4\hat{C}\hat{v}_o^2)$. $\hat{C}$ is a constant that shall be determined later on. The assumption that initially $Q(X) < \infty$ implies trivially that $a < \infty$. Moreover we define

$$U = \sup_{t : 0 \leqslant t \leqslant \hat{C}\hat{v}_o} \; \sup_{i \in A} |v_i(t)| \tag{4.3}$$

where $A$ is the set of the particles of the background that during the time $[0, \hat{C}\hat{v}_o]$ enter in the space interval $[-3\hat{C}\hat{v}_o^2, 3\hat{C}\hat{v}_o^2]$.

The strategy of the proof is the following: we assume that $U \leqslant \frac{\hat{v}_o}{2}$ and until a time $T = \hat{C}\hat{v}_o$, $|\hat{v}(t) - \hat{v}_o|$ remains smaller than $\frac{1}{10}\hat{v}_o$. Then we prove that, for small $\hat{C}$, $U \leqslant \frac{\hat{v}_o}{4}$ and, until $T$, $|\hat{v}(t) - \hat{v}_o|$ remains smaller than $\frac{1}{20}\hat{v}_o$. Hence, by continuity, starting from the initial velocity, these assumptions are actually true. Moreover, if $U \leqslant \frac{\hat{v}_o}{2}$ and $|\hat{v}(t) - \hat{v}_o|$ is smaller than $\frac{1}{10}\hat{v}_o$, then $|\hat{v}(t) - \hat{v}_o|$ must be smaller than a constant, which is the statement of Theorem 4.1.

The main point in the proof is an easy observation on the interaction of the fast particle with a slow one alone in the space. It is well known that the energy and the momentum conservation imply that the scattering process leaves the velocities unchanged. Thus a slow particle can change the velocity of the fast one only if on the slow particle acts an external force (produced by the other particles of the background). This term can be sharply estimated as we shall see.

The proof has three main steps.

(i) Step 1.

Consider the $n$-partial dynamics. We use the estimates of the Appendix and we proceed as in Section 2 (from (2.18) to (2.24)). Bound (2.24) here reads:

$$\max_i \sup_{0 \leqslant \tau \leqslant t} |v_i^n(t)| < C_{15}(\log(e+n))^{\frac{1}{2}} + C_{16} t, \qquad t \geqslant 0 \tag{4.4}$$

$C_{15}, C_{16}$ not depending on $\hat{v}_o$.

(ii) Step 2.

For any particle $i$, we define $n_i^* = \mathrm{Int}[(e+|x_i(0)|)\, e^{\hat{C}\hat{v}_o}]$. Then Eq. (4.4) gives

$$\sup_{0 \leqslant t \leqslant \hat{C}\hat{v}_o} |v_i^{n_i^*}(t)| < C_{17}(\log(e+|x_i(0)|))^{\frac{1}{2}} + C_{18}\hat{C}\hat{v}_o \tag{4.5}$$

where we have used the fact that the square root is a concave function.

Moreover we observe that the infinite dynamics differs from the $n_i^*$ one by a negligible quantity, as we can prove following the steps of the proof of Theorem 2.1 substituting definition (2.27) by

$$\begin{aligned}\delta_i(n, t) = {} & |x_i^n(t) - x_i^{n-1}(t)| + |v_i^n(t) - v_i^{n-1}(t)| + |\hat{x}^n(t) - \hat{x}^{n-1}(t)| \\ & + |\hat{v}^n(t) - \hat{v}^{n-1}(t)|\end{aligned} \tag{4.6}$$

and studying the problem for $t \leqslant \hat{C}\hat{v}_o$.

In conclusion for any $i$,

$$\sup_{0 \leqslant t \leqslant \hat{C}\hat{v}_o} |v_i(t)| < C_{19}(\log(e+|x_i(0)|))^{\frac{1}{2}} + C_{18}\hat{C}\hat{v}_o. \tag{4.7}$$

Hence, if $C_{18}\hat{C} < \frac{1}{5}$ and $\hat{v}_o$ is large enough, only the particles initially in $B(0, 4\hat{C}\hat{v}_o^2)$ could enter in $B(0, 3\hat{C}\hat{v}_o^2)$ during a time $\hat{C}\hat{v}_o$ and interact with the fast particle. Moreover these particles have a velocity smaller than $\frac{\hat{v}_o}{4}$, as we wanted to prove.

(iii) Step 3.

First we study the problem for the $n_o$-partial dynamics with $n_o = \mathrm{Int}[(e+3\hat{C}\hat{v}_o^2)\, e^{\hat{C}\hat{v}_o^2}]$ and then we consider the infinite case. From here to formula (4.16) for simplicity we omit to explicit the dependence on $n_o$.

The result of step 2 tells us that the particles of the background until a time $\hat{C}\hat{v}_o$ ($\hat{C}$ small) move with a speed at most $\frac{1}{2}\hat{v}_o$. This means that there is a gap between the velocity of the fast particle and the velocities of the particles of the background. We show now that in this situation the background cannot slow down the fast particle for a long time. More precisely, in order to study $\hat{v}(t)$ we introduce an other quantity $\hat{p}(t)$, that coincides with $\hat{v}(t)$ when the scattering process is absent and is slowly varying during the collision (as it is suggested by a perturbation theory):

$$\hat{p}(t) = \hat{v}(t) + \sum_{i \in \mathbb{N}} \frac{\hat{\phi}(\hat{x}(t) - x_i(t))}{M(\hat{v}(t) - v_i(t))}. \tag{4.8}$$

By using the Eq. (4.1), we have

$$\begin{aligned}\dot{\hat{p}}(t) &= \sum_{i \in \mathbb{N}} \frac{\hat{\phi}(\hat{x}(t) - x_i(t))}{M(\hat{v}(t) - v_i(t))^2} [\dot{v}_i(t) - \dot{\hat{v}}(t)] \\ &= \sum_{i \in \mathbb{N}} \frac{\hat{\phi}(\hat{x}(t) - x_i(t))}{M(\hat{v}(t) - v_i(t))^2} \{\hat{F}(x_i(t) - \hat{x}(t)) \\ &\quad + \sum_{\substack{j = \mathbb{N} \\ j \neq i}} F(x_i(t) - x_j(t)) - M^{-1} \sum_{j = \mathbb{N}} \hat{F}(\hat{x}(t) - x_j(t))\}.\end{aligned} \tag{4.9}$$

Hence

$$|\dot{\hat{p}}(t)| \leqslant \frac{C_{20}}{\hat{v}_o^2} N^2(t) \tag{4.10}$$

where $N(t)$ is the number of particles contained in the interval of center in $\hat{x}(t)$ and size 4 $\max(r, \hat{r})$. Hence

$$|\hat{p}(\hat{C}\hat{v}_o) - \hat{p}(0)| \leqslant \frac{C_{20}}{\hat{v}_o^2} \int_0^{\hat{C}\hat{v}_o} N^2(t)\, dt. \tag{4.11}$$

Now we observe that the positivity and the smoothness of the potential in the origin implies that $N^2(t)$ is smaller than a constant times the energy of a region $B(\hat{x}(t), 2\max(r, \hat{r}))$:

$$N^2(t) \leqslant C_{21} Q(X(t); \hat{x}(t), \max(r, \hat{r})) \leqslant C_{21} W(X(t), 0, 2\hat{C}\hat{v}_o^2) \leqslant C_{22}\hat{C}\hat{v}_o^2, \tag{4.12}$$

where we have used the result proved in the Appendix. Moreover the particles of the background that contribute to $N(t)$ cannot contribute to $N(t')$

if $|t-t'|\, \hat{v}_o \frac{9}{10} > 2\max(r, \hat{r}) + \frac{1}{2}\hat{v}_o\, |t-t'|$, that is $|t-t'| > C_{23}\hat{v}_o^{-1}$. Hence each particle of the background which interacts with the fast one contributes to the integral $\int_0^{\hat{C}\hat{v}_o} N(t)\, dt$ by an amount smaller than const. $\hat{v}_o^{-1}$. Since the number of these particles is smaller than $C_{24}\hat{C}\hat{v}_o^2$, we have

$$\int_0^{\hat{C}\hat{v}_o} N(t)\, dt \leqslant C_{25}\hat{C}\hat{v}_o. \tag{4.13}$$

Hence, by using (4.12) and (4.13), we have

$$\int_0^{\hat{C}\hat{v}_o} N^2(t)\, dt \leqslant \Big(\sup_{0 \leqslant t \leqslant \hat{C}\hat{v}_o} N(t)\Big) \int_0^{\hat{C}\hat{v}_o} N(t)\, dt \leqslant C_{26}\hat{v}_o^2 \hat{C}^{\frac{3}{2}}. \tag{4.14}$$

So that, from (4.11) it follows

$$|\hat{p}(\hat{C}\hat{v}_o) - \hat{p}(0)| \leqslant C_{27}\hat{C}^{\frac{3}{2}}. \tag{4.15}$$

Finally

$$|\hat{v}(t) - \hat{v}_o| \leqslant |\hat{v}(t) - \hat{p}(t)| + |\hat{p}(t) - \hat{p}(0)| + |\hat{p}(0) - \hat{v}_o|. \tag{4.16}$$

We consider the three terms of the right hand side. The first and the third ones by definition (4.8) and (4.12) are bounded by $C_{28}\hat{C}^{\frac{1}{2}}$ for $0 \leqslant t \leqslant \hat{C}\hat{v}_o$, while the second term is bounded by (4.15).

We conclude that the fast particle almost conserves its velocity, so that the assumption that $\hat{v}(t) \leqslant \frac{9}{10}\hat{v}_o$ is true and then for this partial dynamics the Theorem is valid. Since the difference between this partial dynamics and the infinite one is negligible, Theorem 4.1 is proved. ∎

In the previous Theorem we have proved that there exists a constant $\hat{C}$ such that the result is true. Actually we have proved more:

**Corollary 4.1.** There exist a constant $\tilde{C}$ such that, for any positive $\tau \leqslant \hat{C}$, provided $\hat{v}_o$ is large enough (i.e., larger than a quantity depending on $\tau$), we have:

$$|\hat{v}(\tau\hat{v}_o) - \hat{v}_o| \leqslant \tilde{C}\tau^{\frac{1}{2}}. \tag{4.17}$$

## APPENDIX

We consider the system defined by Eqs. (4.1). The case discussed in Section 2 is recovered by putting $\hat{\phi} = 0$. When $\hat{\phi} \neq 0$, we assume that $U \leqslant \frac{\hat{v}_o}{2}$ and $\hat{v}(t) > \frac{9}{10}\hat{v}_o$. We prove Lemma 2.1.

For $0 \leqslant s \leqslant t \leqslant T$, we define

$$R(n, t, s) = \log(e+n)) + \int_0^t V^n(\tau)\, d\tau + \int_s^t V^n(\tau)\, d\tau \tag{A.1}$$

(note that $R(n, t, t) = R(n, t)$ and $R(n, t, 0) < 2R(n, t)$) and compute the derivative with respect to $s$ of the quantity:

$$W(X^n(s); \mu, R(n, t, s)) = \sum_i f_i^{\mu, R(n, t, s)} \left[ \frac{v_i^2}{2} + \frac{1}{2} \sum_{j \neq i} \phi_{i,j} + \hat{\phi}(x_i - \hat{x}) + 1 \right]. \tag{A.2}$$

We have:

$$\begin{aligned} &\dot{W}(X^n(s); \mu, R(n, t, s)) \\ &\quad = \sum_i f' \left( \frac{|x_i - \mu|}{R(n, t, s)} \right) \left( \frac{\hat{x}_i^\mu \cdot v_i}{R(n, t, s)} - \frac{\dot{R}(n, t, s)}{R^2(n, t, s)} |x_i - \mu| \right) \\ &\quad \times \left[ \frac{v_i^2}{2} + \frac{1}{2} \sum_{j \neq i} \phi_{i,j} + \hat{\phi}(x_i - \hat{x}) + 1 \right] \\ &\quad + \sum_{i \neq j} f_i^{\mu, R(n, t, s)} \left( v_i F_{i,j} - \frac{1}{2} F_{i,j}(v_i - v_j) \right) + \sum_i f_i^{\mu, R(n, t, s)} \hat{v} \hat{F}(x_i - \hat{x}). \end{aligned} \tag{A.3}$$

In (A.3) we neglect the explicit dependence on $s$ (and $n$) of $x_i$ and $v_i$ and denote by $\hat{x}_i^\mu$ the unit vector in the direction of $(x_i - \mu)$.

We note that the first term in the right hand side of (A.3) is not positive. Indeed $f' \leqslant 0$, $|x_i - \mu| > R$, $|v_i| \leqslant V^n(s)$ and $\dot{R}(n, t, s) = -V^n(s)$ so that:

$$\frac{\hat{x}_i^\mu \cdot v_i}{R(n, t, s)} - \frac{\dot{R}(n, t, s)}{R^2(n, t, s)} |x_i - \mu| \geqslant -\frac{|v_i|}{R(n, t, s)} - \frac{\dot{R}(n, t, s)}{R(n, t, s)} \geqslant 0. \tag{A.4}$$

On the other hand the second term is (by using $F_{i,j} = -F_{j,i}$):

$$\tfrac{1}{2} \sum_{i \neq j} f_i^{\mu, R(n, t, s)} (F_{i,j} \cdot (v_i + v_j)) = \tfrac{1}{2} \sum_{i \neq j} (f_i^{\mu, R(n, t, s)} - f_j^{\mu, R(n, t, s)})(F_{i,j} \cdot v_i). \tag{A.5}$$

By the obvious inequality:

$$|f_i^{\mu, R(n, t, s)} - f_j^{\mu, R(n, t, s)}| \leqslant 2R(n, t, s)^{-1} |x_i - x_j|, \tag{A.6}$$

the modulus of the quantity in (A.5) is bounded by

$$-C_{29}\frac{\dot{R}(n,t,s)}{R(n,t,s)}\sum_{i\neq j}\chi(|x_i-x_j|<r)$$
$$\times\chi(|x_i-\mu|<2R(n,t,s)+r)\,\chi(|x_j-\mu|<2R(n,t,s)+r)$$
$$\leqslant -C_{29}\frac{\dot{R}(n,t,s)}{R(n,t,s)}\sum_{i\neq j}\chi(|x_i-x_j|<r)$$
$$\times\chi(|x_i-\mu|<4R(n,t,s))\,\chi(|x_j-\mu|<4R(n,t,s)), \tag{A.7}$$

where from now on we have chosen $R\geqslant 2\max(r,\hat{r})+1$. This is possible without loss of generality because we are looking for properties valid for large $R$.

The superstability of the interaction (i.e., $\phi(x)\geqslant 0$, $\phi(0)>0$) implies

$$\sum_{i\neq j}\chi(|x_i-x_j|<r)\,\chi(|x_i-\mu|<4R(n,t,s))\,\chi(|x_j-\mu|<4R(n,t,s))$$
$$\leqslant \text{const. } W(X^n(s);\mu,4R(n,t,s)). \tag{A.8}$$

For the proof of (A.8) see [1, Lemma 2.1(iii)].

Hence

$$\dot{W}(X^n(s);\mu,R(n,t,s))$$
$$\leqslant -C_{30}\frac{\dot{R}(n,t,s)}{R(n,t,s)}W(X^n(s);\mu,4R(n,t,s))+\sum_i f_i^{\mu,R(n,t,s)}\,|\hat{v}\hat{F}(x_i-\hat{x})|. \tag{A.9}$$

We state a property of $W$. We observe that

$$W(X;\mu,2R)\leqslant\sum_{|n|\leqslant 2}W(X;\mu+2nR,R) \tag{A.10}$$

because any term of the left hand side is equal or bounded by a term of the right hand side. Moreover the other terms of this side are positive.

Setting:

$$W(X;R)=\sup_{\mu\in\mathbb{R}}W(X;\mu,R), \tag{A.11}$$

we have

$$W(X;\mu,2R)\leqslant 5W(X,R) \tag{A.12}$$

From (A.12), the differential inequality (A.9) becomes

$$\dot{W}(X^n(s); \mu, R(n, t, s)) \leqslant -C_{31} \frac{\dot{R}(n, t, s)}{R(n, t, s)} W(X^n(s); R(n, t, s)) + \sum_i f_i^{\mu, R(t, s)} \, |\hat{v}\hat{F}(x_i - \hat{x})|, \tag{A.13}$$

so that

$$\begin{aligned} W(X^n(s); \mu, R(n, t, s)) &\leqslant W(X^n(0); R(n, t, 0)) + \left[ -C_{31} \int_0^s d\tau \frac{\dot{R}(n, t, \tau)}{R(n, t, \tau)} W(X^n(\tau); R(n, t, \tau)) \right] \\ &+ \int_0^s d\tau \sum_i f_i^{\mu, R(n, t, \tau)} \, |\hat{v}\hat{F}(x_i - \hat{x})|]. \end{aligned} \tag{A.14}$$

We observe that in the last integral contribute only particles that initially are in the interval $B(\mu, 4R(n, t, 0))$. The number of these particles is smaller than $W(X^n(0); 4R(n, t, 0))$. Moreover the fast particle interacts with the particle $x_i$ for a time proportional to $\hat{v}_o^{-1}$ so that

$$\begin{aligned} &\int_0^s d\tau \sum_i f_i^{\mu, R(n, t, \tau)} |\hat{v}\hat{F}(x_i - \hat{x})| \\ &\quad \leqslant C_{32} W(X^n(0), \mu; 4R(n, t, 0)) \leqslant C_{33} W(X^n(0); R(n, t, 0)), \end{aligned} \tag{A.15}$$

where in the last estimate we have used (A.12). We put (A.15) in (A.14), we take the *sup* on $\mu$, and we solve the differential inequality. We have for $s \leqslant t$:

$$W(X^n(s); R(n, t, s)) \leqslant C_{34} W(X^n(0); R(n, t, 0)) \left( \frac{R(n, t, 0)}{R(n, t, s)} \right)^{C_{31}}. \tag{A.16}$$

Since $\frac{R(t, 0)}{R(t, s)} < 2$, we conclude that:

$$\begin{aligned} W(X^n(t); R(n, t)) &\leqslant C_{35} W(X^n(0); R(n, t, 0)) \leqslant C_{36} Q(X) \, R(n, t) \\ &\leqslant C_{37} R(n, t). \quad \blacksquare \end{aligned} \tag{A.17}$$

## REFERENCES

1. E. Caglioti, C. Marchioro, and M. Pulvirenti, Non-equilibrium dynamics of three-dimensional infinite particle systems, *Comm. Math. Phys.* **215**:25–43 (2000).
2. R. L. Dobrushin and J. Fritz, Non-equilibrium dynamics of one-dimensional infinite particle systems with a hard-core interaction, *Comm. Math. Phys.* **55**:275–292 (1977).
3. J. Fritz and R. L. Dobrushin, Non-equilibrium dynamics of two-dimensional infinite particle systems with a singular interaction, *Comm. Math. Phys.* **57**:67–81 (1977).
4. O. E. Lanford, Classical mechanics of one-dimensional systems with infinitely many particles. I. An existence theorem, *Comm. Math. Phys.* **9**:176–191 (1968).
5. O. E. Lanford, Classical mechanics of one-dimensional systems with infinitely many particles. II. Kinetic Theory, *Comm. Math. Phys.* **11**:257–292 (1969).
6. C. Marchioro, A. Pellegrinotti, and M. Pulvirenti, Remarks on the existence of non-equilibrium dynamics, Proceedings Esztergom Summer School, *Coll. Math. Soc. Janos Bolyai* **27**:733–746 (1978).
7. C. Marchioro and M. Pulvirenti, Time evolution of infinite one-dimensional Coulomb systems, *J. Statist. Phys.* **27**:809–822 (1982).
8. Ya. G. Sinai and M. Solovievich, One-dimensional classical massive particle in the ideal gas, *Comm. Math. Phys.* **104**:423–443 (1986).